THE INTEGRALS IN GRADSHTEYN AND RYZHIK. PART 3: COMBINATIONS OF LOGARITHMS AND EXPONENTIALS.

VICTOR H. MOLL

ABSTRACT. We present the evaluation of a family of exponential-logarithmic integrals. These have integrands of the form $P(e^{tx}, \ln x)$ where P is a polynomial. The examples presented here appear in sections 4.33, 4.34 and 4.35 in the classical table of integrals by I. Gradshteyn and I. Ryzhik.

1. INTRODUCTION

This is the third in a series of papers dealing with the evaluation of definite integrals in the table of Gradshteyn and Ryzhik [2]. We consider here problems of the form

$$\int_0^\infty e^{-tx}\, P(\ln x)\, dx, \tag{1.1}$$

where $t > 0$ is a parameter and P is a polynomial. In future work we deal with the finite interval case

$$\int_a^b e^{-tx}\, P(\ln x)\, dx, \tag{1.2}$$

where a, $b \in \mathbb{R}^+$ with $a < b$ and $t \in \mathbb{R}$. The classical example

$$\int_0^\infty e^{-x} \ln x\, dx = -\gamma, \tag{1.3}$$

where γ is Euler's constant is part of this family. The integrals of type (1.1) are linear combinations of

$$J_n(t) := \int_0^\infty e^{-tx}\, (\ln x)^n \; dx. \tag{1.4}$$

The values of these integrals are expressed in terms of the gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx \tag{1.5}$$

and its derivatives.

Date: February 1, 2008.
1991 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Integrals.

2. The evaluation

In this section we consider the value of $J_n(t)$ defined in (1.4). The change of variables $s = tx$ yields

$$J_n(t) = \frac{1}{t}\int_0^\infty e^{-s}\,(\ln s - \ln t)^n\;ds. \tag{2.1}$$

Expanding the power yields J_n as a linear combination of

$$I_m := \int_0^\infty e^{-x}\,(\ln x)^m\;dx, \quad 0 \leq m \leq n. \tag{2.2}$$

An analytic expression for these integrals can be obtained directly from the representation of the *gamma function* in (1.5).

Proposition 2.1. For $n \in \mathbb{N}$ we have

$$\int_0^\infty (\ln x)^n\;x^{s-1}e^{-x}\,dx = \left(\frac{d}{ds}\right)^n \Gamma(s). \tag{2.3}$$

In particular

$$I_n := \int_0^\infty (\ln x)^n\;e^{-x}\,dx = \Gamma^{(n)}(1). \tag{2.4}$$

Proof. Differentiate (1.5) n-times with respect to the parameter s. □

Example 2.2. Formula **4.331.1** in [2] states that[1]

$$\int_0^\infty e^{-\mu x}\;\ln x\,dx = -\frac{\delta}{\mu} \tag{2.5}$$

where $\delta = \gamma + \ln\mu$. This value follows directly by the change of variables $s = \mu x$ and the classical special value $\Gamma'(1) = -\gamma$. The reader will find in chapter 9 of [1] details on this constant. In particular, if $\mu = 1$, then $\delta = \gamma$ and we obtain (1.3):

$$\int_0^\infty e^{-x}\;\ln x\,dx = -\gamma. \tag{2.6}$$

The change of variables $x = e^{-t}$ yields the form

$$\int_{-\infty}^\infty t\,e^{-t}\,e^{-e^{-t}}\;dt = \gamma. \tag{2.7}$$

Many of the evaluations are given in terms of the *polygamma function*

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x). \tag{2.8}$$

Properties of ψ are summarized in Chapter 1 of [4]. A simple representation is

$$\psi(x) = \lim_{n\to\infty}\left(\ln n - \sum_{k=0}^n \frac{1}{x+k}\right), \tag{2.9}$$

from where we conclude that

$$\psi(1) = \lim_{n\to\infty}\left(\ln n - \sum_{k=1}^n \frac{1}{k}\right) = -\gamma, \tag{2.10}$$

[1] The table uses C for the Euler constant.

this being the most common definition of the Euler's constant γ. This is precisely the identity $\Gamma'(1) = -\gamma$.

The derivatives of ψ satisfy

$$\psi^{(m)}(x) = (-1)^{m+1} m!\, \zeta(m+1, x), \tag{2.11}$$

where

$$\zeta(z, q) := \sum_{n=0}^{\infty} \frac{1}{(n+q)^z} \tag{2.12}$$

is the *Hurwitz zeta function*. This function appeared in [3] in the evaluation of some logarithmic integrals.

Example 2.3. Formula **4.335.1** in [2] states that

$$\int_0^\infty e^{-\mu x} \left(\ln x\right)^2 \, dx = \frac{1}{\mu} \left[\frac{\pi^2}{6} + \delta^2 \right], \tag{2.13}$$

where $\delta = \gamma + \ln \mu$ as before. This can be verified using the procedure described above: the change of variable $s = \mu x$ yields

$$\int_0^\infty e^{-\mu x} \left(\ln x\right)^2 \, dx = \frac{1}{\mu} \left(I_2 - 2 I_1 \ln \mu + I_0 \ln^2 \mu \right), \tag{2.14}$$

where I_n is defined in (2.4). To complete the evaluation we need some special values: $\Gamma(1) = 1$ is elementary, $\Gamma'(1) = \psi(1) = -\gamma$ appeared above and using (2.11) we have

$$\psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \left(\frac{\Gamma'(x)}{\Gamma(x)} \right)^2. \tag{2.15}$$

The value

$$\psi'(1) = \zeta(2) = \frac{\pi^2}{6}, \tag{2.16}$$

where $\zeta(z) = \zeta(z, 1)$ is the Riemann zeta function, comes directly from (2.11). Thus

$$\Gamma''(1) = \zeta(2) + \gamma^2. \tag{2.17}$$

Let $\mu = 1$ in (2.13) to produce

$$\int_0^\infty e^{-x} \left(\ln x\right)^2 \, dx = \zeta(2) + \gamma^2. \tag{2.18}$$

Similar arguments yields formula **4.335.3** in [2]:

$$\int_0^\infty e^{-\mu x} \left(\ln x\right)^3 \, dx = -\frac{1}{\mu} \left[\delta^3 + \tfrac{1}{2}\pi^2 \delta - \psi''(1) \right], \tag{2.19}$$

where, as usual, $\delta = \gamma + \ln \mu$. The special case $\mu = 1$ now yields

$$\int_0^\infty e^{-x} \left(\ln x\right)^3 \, dx = -\gamma^3 - \tfrac{1}{2}\pi^2 \gamma + \psi''(1). \tag{2.20}$$

Using the evaluation

$$\psi''(1) = -2\zeta(3) \tag{2.21}$$

produces

$$\int_0^\infty e^{-x} \left(\ln x\right)^3 \, dx = -\gamma^3 - \tfrac{1}{2}\pi^2 \gamma - 2\zeta(3). \tag{2.22}$$

Problem 2.4. In [1], page 203, we introduced the notion of *weight* for some real numbers. In particular, we have assigned $\zeta(j)$ the weight j. Differentiation increases the weight by 1, so that $\zeta'(3)$ has weight 4. The task is to check that the integral

$$I_n := \int_0^\infty e^{-x} (\ln x)^n \, dx \tag{2.23}$$

is a homogeneous form of weight n.

3. A small variation

Similar arguments are now employed to produce a larger family of integrals. The representation

$$\int_0^\infty x^{s-1} e^{-\mu x} \, dx = \mu^{-s} \Gamma(s), \tag{3.1}$$

is differentiated n times with respect to the parameter s to produce

$$\int_0^\infty (\ln x)^n \, x^{s-1} e^{-\mu x} \, dx = \left(\frac{d}{ds} \right)^n \left[\mu^{-s} \Gamma(s) \right]. \tag{3.2}$$

The special case $n = 1$ yields

$$\begin{aligned} \int_0^\infty x^{s-1} e^{-\mu x} \, \ln x \, dx &= \frac{d}{ds} \left[\mu^{-s} \Gamma(s) \right] \\ &= \mu^{-s} \left(\Gamma'(s) - \ln \mu \, \Gamma(s) \right) \\ &= \mu^{-s} \Gamma(s) \left(\psi(s) - \ln \mu \right). \end{aligned} \tag{3.3}$$

This evaluation appears as **4.352.1** in [2]. The special case $\mu = 1$ yields

$$\int_0^\infty x^{s-1} e^{-x} \, \ln x \, dx = \Gamma'(s), \tag{3.4}$$

that is **4.352.4** in [2].

Special values of the gamma function and its derivatives yield more concrete evaluations. For example, the functional equation

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{3.5}$$

that is a direct consequence of $\Gamma(x+1) = x\Gamma(x)$, yields

$$\psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k}. \tag{3.6}$$

Replacing $s = n+1$ in (3.3) we obtain

$$\int_0^\infty x^n e^{-\mu x} \, \ln x \, dx = \frac{n!}{\mu^{n+1}} \left(\sum_{k=1}^n \frac{1}{k} - \gamma - \ln \mu \right), \tag{3.7}$$

that is **4.352.2** in [2].

The final formula of Section **4.352** in [2] is **4.352.3**

$$\int_0^\infty x^{n-1/2} e^{-\mu x} \, \ln x \, dx = \frac{\sqrt{\pi} \, (2n-1)!!}{2^n \, \mu^{n+1/2}} \left[2 \sum_{k=1}^n \frac{1}{2k-1} - \gamma - \ln(4\mu) \right].$$

This can also be obtained from (3.3) by using the classical values

$$\begin{aligned} \Gamma(n+\tfrac{1}{2}) &= \frac{\sqrt{\pi}}{2^n}(2n-1)!! \\ \psi(n+\tfrac{1}{2}) &= -\gamma + 2\left(\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2\right). \end{aligned}$$

The details are left to the reader.

Section **4.353** of [2] contains three peculiar combinations of integrands. The first two of them can be verified by the methods described above: formula **4.353.1** states

$$\int_0^\infty (x-\nu)x^{\nu-1}e^{-x} \ln x \, dx = \Gamma(\nu), \tag{3.8}$$

and **4.353.2** is

$$\int_0^\infty (\mu x - n - \tfrac{1}{2})x^{n-\frac{1}{2}}e^{-\mu x} \ln x \, dx = \frac{(2n-1)!!}{(2\mu)^n}\sqrt{\frac{\pi}{\mu}}. \tag{3.9}$$

Acknowledgments. The author wishes to thank Luis Medina for a careful reading of an earlier version of the paper. The partial support of NSF-DMS 0409968 is also acknowledged.

References

[1] G. Boros and V. Moll. *Irresistible Integrals*. Cambridge University Press, New York, 1st edition, 2004.
[2] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.
[3] V. Moll. The integrals in Gradshteyn and Ryzhik. Part 1: a family of logarithmic integrals. *Scientia*, 13:1–8, 2006.
[4] H. M. Srivastava and J. Choi. *Series associated with the zeta and related functions*. Kluwer Academic Publishers, 1st edition, 2001.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu